Novikov structures on solvable Lie algebras

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Abstract

We study Novikov algebras and Novikov structures on finite-dimensional Lie algebras. We show that a Lie algebra admitting a Novikov structure must be solvable. Conversely we present an example of a nilpotent two-step solvable Lie algebra without any Novikov structure. We construct Novikov structures on certain Lie algebras via classical r-matrices and via extensions. In the latter case we lift Novikov structures on an abelian Lie algebra $\mathfrak{a}$ and a Lie algebra $\mathfrak{b}$ to certain extensions of $\mathfrak{b}$ by $\mathfrak{a}$. We apply this to prove the existence of affine and Novikov structures on several classes of two-step solvable Lie algebras. In particular we generalize a well known result of Scheuneman concerning affine structures on three-step nilpotent Lie algebras.
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1. Introduction

Novikov algebras arise in several contexts in mathematics and physics. They were firstly introduced in the study of Hamiltonian operators concerning integrability of certain nonlinear partial differential equations, see [1]. They also appear in connection with Poisson brackets of hydrodynamic type [3], and operator Yang-Baxter equations [12]. It is also well known that there are connections between Novikov algebras and vertex algebras. In section 6 of [2], for example, it is shown that the left-symmetry is one of the axioms of a vertex algebra. In fact, a Novikov algebra is a special case of a left-symmetric algebra (LSA). LSAs are also of interest in conformal

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field theory, see [8,9]: if one studies perturbations of two-dimensional conformal field theories by holomorphic fields, then the genus one partition function is controlled by a contact term left-symmetric algebra given in terms of the operator product expansion. This has applications to vertex operator algebras, two-dimensional QCD, topological strings and modular properties of generalized characters of chiral algebras.

LSAs also arise in the study of affine manifolds, affine crystallographic groups, convex homogeneous cones, and the areas mentioned above. An LSA is a Lie-admissible algebra: the commutator defines a Lie algebra. One may ask which Lie algebras can arise that way. This is the existence question of affine structures on Lie algebras. It is quite difficult and has been studied a lot, see [5]. For Novikov structures the existence question is more accessible. We are able to prove interesting results. On the other hand we study natural construction methods of left-symmetric and Novikov algebras. We use classical and generalized r-matrices to obtain left-symmetric and Novikov structures on Lie algebras. Another construction of affine structures is the lifting of given affine structures on an abelian Lie algebra $\mathfrak{a}$ and a Lie algebra $\mathfrak{b}$ to an affine structure on the extension of $\mathfrak{b}$ by $\mathfrak{a}$. We describe the conditions for lifting affine and Novikov structures.

The paper is organized as follows: in the first section we introduce affine and Novikov structures and show that a Lie algebra admitting a Novikov structure must be solvable. Then we give an example of an eight-dimensional four-step nilpotent and two-step solvable Lie algebra which does not admit any Novikov structure. In the second section we show how to construct Novikov structures on certain Lie algebras via classical r-matrices. In the third section we determine the conditions for lifting affine and Novikov structures via extensions. We apply this to give a new proof of Scheunemann's result concerning affine structures on three-step nilpotent Lie algebras. For Novikov structures we obtain a similar result, if the Lie algebra is two- or three-generated. However, the proof does not work in general for Novikov structures. But for a certain class of two-step solvable Lie algebras we are able to obtain Novikov structures via extensions. Among them is the class of filiform Lie algebras with abelian commutator algebra.

In the last section we show by cohomological considerations that the lifting procedure via extensions can be essentially reduced to nilpotent Lie algebras. As an application we can prove a generalization of Scheuneman's result to certain two-step solvable Lie algebras.

2. Affine and Novikov structures

Let k be a field of characteristic zero. A Novikov algebra and, more generally, an LSA is defined as follows:

Definition 2.1. An algebra $(A, \cdot)$ over k with product $(x, y) \mapsto x \cdot y$ is called *left-symmetric algebra (LSA)*, if the product is left-symmetric, i.e. if the identity:

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \tag{1}$$

is satisfied for all $x, y, z \in A$. The algebra is called *Novikov*, if in addition

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{2}$$

is satisfied.

Denote by $L(x), R(x)$ the left respectively right multiplication operator in the algebra $(A, \cdot)$. Then a LSA is a Novikov algebra if the right multiplications commute:

$$[R(x), R(y)] = 0.$$

It is well known that LSAs are Lie-admissible algebras: the commutator defines a Lie bracket. The associated Lie algebra then is said to admit a left-symmetric structure, or affine structure.

Definition 2.2. An *affine structure* on a Lie algebra $\mathfrak{g}$ over k is a left-symmetric product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

$$[x, y] = x \cdot y - y \cdot x \tag{3}$$

for all $x, y, z \in \mathfrak{g}$. If the product is Novikov, we say that $\mathfrak{g}$ admits a *Novikov structure*.

An affine structure on a Lie algebra $\mathfrak{g}$ corresponds to a left-invariant affine structure on a connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$. Such structures play an important role for affine crystallographic groups and affine manifolds, see [5]. It is well known that not every Lie algebra admits an affine structure. A finite-dimensional Lie algebra $\mathfrak{g}$ over k with $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ does not admit an affine structure. But also not every solvable Lie algebra admits an affine structure. There exist complicated examples of even nilpotent Lie algebras which do not admit an affine structure, see [4,5]. These examples are of dimension 10, 11, 12, 13. On the other hand there are many classes of solvable and nilpotent Lie algebras which do admit an affine structure: every positively graded Lie algebra admits an affine structure, and every two and three-step nilpotent Lie algebra admits an affine structure.

We need some notations for solvable and nilpotent Lie algebras. Denote the terms of the commutator series by $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$. Then $\mathfrak{g}$ is called p-step solvable if $\mathfrak{g}^{(p+1)} = 0$. Denote the terms of the lower central series by $\mathfrak{g}^1 = \mathfrak{g}$, $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$. The Lie algebra $\mathfrak{g}$ is called p-step nilpotent, if $\mathfrak{g}^{p+1} = 0$.

Remark 2.3. A three-step nilpotent Lie algebra is two-step solvable: $\mathfrak{g}^4 = 0$ and the Jacobi identity implies $\mathfrak{g}^{(3)} = 0$, since:

$$[[x, y], [z, w]] = [w, [z, [x, y]]] - [z, [w, [x, y]]] = 0.$$

The following two lemmas are elementary but useful. We leave the proof to the reader.

Lemma 2.4. *Let $(A, \cdot)$ be an algebra product on the vector space of a Lie algebra $\mathfrak{g}$. Then $(A, \cdot)$ defines a Novikov structure on $\mathfrak{g}$ if and only if for all $x, y \in \mathfrak{g}$:*

$$L(x) - R(x) = \operatorname{ad}(x), \qquad [L(x), L(y)] = L([x, y]), \qquad [R(x), R(y)] = 0$$

Lemma 2.5. *Let $(A, \cdot)$ be a Novikov algebra. Then the following two identities hold for all $x, y, z \in A$:*

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0, \qquad x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0.$$

Definition 2.6. An affine structure on a Lie algebra $\mathfrak{g}$ is called *complete*, if the right multiplications $R(x)$ are nilpotent for all $x \in \mathfrak{g}$.

The existence question of affine structures is very difficult in general. For Novikov algebras this question is more accessible. First, it follows from a result in [13] that the underlying Lie algebra of a Novikov algebra is solvable.

Proposition 2.7. *Any finite-dimensional Lie algebra over k admitting a Novikov structure is solvable.*

Proof. Assume that $\mathfrak{g}$ admits a Novikov structure given by $(x, y) \mapsto x \cdot y$. If $\bar{k}$ denotes the algebraic closure of k, then this Novikov structure extends to a Novikov structure on the Lie algebra

$\mathfrak{g} \otimes \bar{k}$ over $\bar{k}$. Therefore, there is no loss in generality in assuming that k itself is algebraically closed. Denote by $R(x)$ the right multiplication in the Novikov algebra A, i.e. $R(x)(y) = y \cdot x$. The algebra A is called right-nilpotent if $R_A = \{R(x) | x \in A\}$ satisfies $R_A^n = 0$ for some $n \geq 1$. Let I, J be two right-nilpotent ideals in A. Since $[R(x), R(y)] = 0$ the sum $I + J$ is also a right-nilpotent ideal. Since A is finite-dimensional there exists a largest right-nilpotent ideal of A, denoted by $N(A)$. Now $N(A)$ is a complete left-symmetric algebra since its right multiplications are nilpotent. It is known that the Lie algebra of a complete LSA is solvable, see [11]. Hence the Lie algebra $\mathfrak{h}$ of $N(A)$ is solvable. On the other hand $A/N(A)$ is a direct sum of fields. This was proved in [13]. It follows that the Lie algebra of $A/N(A)$ is abelian. Hence $\mathfrak{g}/\mathfrak{h}$ is abelian, and $\mathfrak{h}$ is solvable. It follows that $\mathfrak{g}$ is solvable. $\square$

Proposition 2.8. *Let $\mathfrak{g}$ be a finite-dimensional two-step nilpotent Lie algebra. Then $\mathfrak{g}$ admits a Novikov structure.*

Proof. It is well known that for $x, y \in \mathfrak{g}$ the formula $x \cdot y = \frac{1}{2}[x, y]$ defines a left-symmetric structure on $\mathfrak{g}$. It satisfies $x \cdot (y \cdot z) = 0$ for all $x, y, z \in \mathfrak{g}$. Hence the product is also Novikov. $\square$

If we want to to find an example of a solvable or nilpotent Lie algebra without any affine structure, we have to solve nonlinear equations in the entries of the left or right multiplications. This is quite difficult in general. For a Novikov structure the situation is better. The following relation follows from the identities in Lemma 2.4:

$$L([x, y]) + \mathrm{ad}([x, y]) - [\mathrm{ad}(x), L(y)] - [L(x), \mathrm{ad}(y)] = 0. \tag{4}$$

In fact, we have $0 = [R(x), R(y)] = [L(x) - \mathrm{ad}x, L(y) - \mathrm{ad}y]$. Using the fact that L and ad are Lie algebra representations the relation follows. It yields *linear* equations in the entries of the left multiplications which are very helpful for solving the other non-linear equations. It enables us to prove the non-existence of Novikov structures on the following two-step solvable Lie algebra:

Example 2.9. Let $\mathfrak{g}$ be the free four-step nilpotent Lie algebra on two generators x_1 and x_2. Let $(x_1, \ldots, x_8)$ be a basis of $\mathfrak{g}$ with the following Lie brackets:

$$x_3 = [x_1, x_2], \qquad x_4 = [x_1, [x_1, x_2]] = [x_1, x_3],$$
$$x_5 = [x_2, [x_1, x_2]] = [x_2, x_3], \qquad x_6 = [x_1, [x_1, [x_1, x_2]]] = [x_1, x_4],$$
$$x_7 = [x_2, [x_1, [x_1, x_2]]] = [x_2, x_4] = [x_1, [x_2, [x_1, x_2]]] = [x_1, x_5],$$
$$x_8 = [x_2, [x_2, [x_1, x_2]]] = [x_2, x_5]$$

Proposition 2.10. *The two-step solvable Lie algebra from the above example does not admit any Novikov structure.*

Proof. Indeed, the linear equations arising from the above relation, for a Novikov product with this Lie algebra, reduce the system of all equations to very few equations—which are contradictory. Note that $\mathfrak{g}$ admits an affine structure since it is positively graded. $\square$

3. Construction of Novikov structures via classical r-matrices

Classical r-matrices arise in the study of the classical Yang-Baxter equation, differential Lie algebras and Poisson brackets. It was first defined in [12]. We will consider a more general

situation. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{u}$ be a $\mathfrak{g}$-module and $T : \mathfrak{u} \to \mathfrak{g}$ be a linear map. Then we make $\mathfrak{u}$ into an algebra by defining a skew-symmetric product $[\,,\,]_T$ by:

$$[u, v]_T = T(u).v - T(v).u.$$

Definition 3.1. The linear operator $T : \mathfrak{u} \to \mathfrak{g}$ is called a (generalized) *classical r-matrix*, or *T*-operator, if T is a homomorphism of algebras:

$$T([u, v]_T) = [T(u), T(v)]. \tag{5}$$

This is equivalent to the fact, that T satisfies the classical Yang-Baxter equation (CYBE)

$$T(T(u).v - T(v).u) = [T(u), T(v)]. \tag{6}$$

Let $\mathfrak{u} = \mathfrak{g}$ be the adjoint module. Then we obtain the original definition of a classical *r*-matrix and the CYBE of [12]. The CYBE is given in that case by:

$$T([T(x), y] + [x, T(y)]) = [T(x), T(y)].$$

If T is a solution of CYBE, we can use T to construct affine and Novikov structures on Lie algebras.

Theorem 3.2. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{u}$ be a $\mathfrak{g}$-module and $T : \mathfrak{u} \to \mathfrak{g}$ be a linear map satisfying the CYBE. Then the product*:

$$u \circ v = T(u).v \tag{7}$$

is left-symmetric. Hence $\mathfrak{u}_T = (\mathfrak{u}, [\,,\,]_T)$ is a Lie algebra. Then $\mathfrak{u}_T$ admits an affine structure, and $T : \mathfrak{u}_T \to \mathfrak{g}$ is a Lie algebra homomorphism.

Proof. It is easy to see that the product (7) is left-symmetric. Hence the commutator:

$$u \circ v - v \circ u = T(u).v - T(v).u = [u, v]_T$$

automatically is a Lie bracket. This is well known for $\mathfrak{u}$ being the adjoint module. □

Corollary 3.3. *Suppose that $T : \mathfrak{u} \to \mathfrak{g}$ satisfies the CYBE and the condition:*

$$T(T(u).v).w = T(T(u).w).v \tag{8}$$

for all $u, v, w \in \mathfrak{u}$. Then the product (7) defines a Novikov structure on $\mathfrak{u}_T$.

Example 3.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the basis (x, y, h) satisfying $[x, y] = h$, $[x, h] = -2x$, $[y, h] = 2y$. Let $\mathfrak{u} = \mathfrak{g}$ be the adjoint module. Then:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a Novikov structure on the solvable Lie algebra $\mathfrak{u}_T = \mathfrak{g}_T \cong \mathfrak{r}_{3,-1}(\mathbb{C})$.

The Lie brackts of $\mathfrak{u}_T$ are given by:

$$[x, h]_T = -2x \qquad [y, h]_T = 2y \quad \text{and} \quad [x, y]_T = 0.$$

Suppose that $\mathfrak{u} = \mathfrak{g}$ is the adjoint module. How does the structure of $\mathfrak{g}_T$ depends on the structure of $\mathfrak{g}$? We will show that the solvability degree of $\mathfrak{g}_T$ cannot be higher than that of $\mathfrak{g}$. The same applies for nilpotency.

Lemma 3.5. *If we denote by* $\mathrm{Ad}x = \mathrm{ad}T(x) + \mathrm{ad}x \circ T$ *the adjoint operator in* $\mathrm{End}(\mathfrak{g}_T)$, *i.e.,* $\mathrm{Ad}x(y) = [x, y]_T$, *then we have the following identity*:

$$\mathrm{Ad}x_1 \circ \mathrm{Ad}x_2 \circ \cdots \circ \mathrm{Ad}x_n$$

$$= \mathrm{ad}T(x_1) \circ \mathrm{ad}T(x_2) \circ \cdots \circ \mathrm{ad}T(x_n) + \left(\sum_{i=1}^{n} \mathrm{ad}T(x_1) \circ \cdots \circ \mathrm{ad}x_i \circ \cdots \circ \mathrm{ad}T(x_n) \right) \circ T$$

Proof. The formula follows by induction. For $n = 1$ we have $\mathrm{Ad}x_1 = \mathrm{ad}T(x_1) + \mathrm{ad}x_1 \circ T$ by definition. The step from n to $n + 1$ follows by using:

$$T \circ \mathrm{ad}T(x_{n+1}) + T \circ \mathrm{ad}x_{n+1} \circ T = \mathrm{ad}T(x_{n+1}) \circ T$$

$\square$

Corollary 3.6. *If* $\mathfrak{g}$ *is r-step nilpotent, then* $\mathfrak{g}_T$ *is s-step nilpotent with* $s \leq r$.

Proof. By assumption, $\mathrm{ad}x_1 \circ \cdots \circ \mathrm{ad}x_r = 0$ for all $x_1, \ldots, x_r \in \mathfrak{g}$. Then the formula in the above lemma implies $\mathrm{Ad}x_1 \circ \cdots \circ \mathrm{Ad}x_r = 0$. $\square$

Corollary 3.7. *If* $\mathfrak{g}$ *is r-step solvable, then* $\mathfrak{g}_T$ *is s-step solvable with* $s \leq r$.

Proof. This follows again from the formula. We leave this to the reader. Instead we show directly that $\mathfrak{g}_T^{(n)} \subseteq \mathfrak{g}^{(n)}$ for all $n \geq 1$. The case $n = 1$ is obvious. Assume that $\mathfrak{g}_T^{(n-1)} \subseteq \mathfrak{g}^{(n-1)}$. The space $\mathfrak{g}_T^{(n)}$ is spanned by elements of the form $[x, y]_T$ with $x, y \in \mathfrak{g}_T^{(n-1)}$. We have $T(x), T(y) \in \mathfrak{g}^{(n-1)}$, so that:

$$[x, y]_T = [T(x), y] + [x, T(y)] \in [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] = \mathfrak{g}^{(n)}.$$

$\square$

Remark 3.8. The above corollaries need not be true for $\mathfrak{g}$-modules $\mathfrak{u}$ with $\dim \mathfrak{u} = \dim \mathfrak{g}$ which are different from the adjoint module. We mention that it is possible to equip all solvable Lie algebras of dimension $n \leq 3$ over $\mathbb{C}$ with a Novikov structure via classical r-matrices with $\mathfrak{u} = \mathfrak{g}$. Let A be a given LSA with underlying Lie algebra $\mathfrak{g}$. The left-multiplication in A defines a $\mathfrak{g}$-module $\mathfrak{g}_L$. Let id : $\mathfrak{g}_L \to \mathfrak{g}$ be the the identity map. Then $T = \mathrm{id}$ satisfies the CYBE:

$$\mathrm{id}(\mathrm{id}(x).y - \mathrm{id}(y).x) = x.y - y.x = [x, y] = [\mathrm{id}(x), \mathrm{id}(y)].$$

In other words, any LSA-structure on a Lie algebra $\mathfrak{g}$ arises by a (generalized) r-matrix.

4. Construction of Novikov structures via extensions

In the following we will consider Lie algebras $\mathfrak{g}$ which are an extension of a Lie algebra $\mathfrak{b}$ by an abelian Lie algebra $\mathfrak{a}$. Hence we have a short exact sequence of Lie algebras:

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0$$

Since $\mathfrak{a}$ is abelian, there exists a natural $\mathfrak{b}$-module structure on $\mathfrak{a}$. We denote the action of $\mathfrak{b}$ on $\mathfrak{a}$ by $(x, a) \mapsto \varphi(x)a$, where $\varphi : \mathfrak{b} \to \mathrm{End}(\mathfrak{a})$ is the corresponding Lie algebra representation. We have:

$$\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) \tag{9}$$

for all $x, y \in \mathfrak{b}$. Let $\Omega \in Z^2(\mathfrak{b}, \mathfrak{a})$ be a two-cocycle. This means that $\Omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}$ is a skew-symmetric bilinear map satisfying:

$$\varphi(x)\Omega(y, z) - \varphi(y)\Omega(x, z) + \varphi(z)\Omega(x, y) = \Omega([x, y], z) - \Omega([x, z], y) + \Omega([y, z], x)$$

(10)

We obtain a Lie bracket on $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ by:

$$[(a, x), (b, y)] := (\varphi(x)b - \varphi(y)a + \Omega(x, y), [x, y])$$

(11)

for $a, b \in \mathfrak{a}$ and $x, y \in \mathfrak{b}$. As a special case we obtain the two-step solvable Lie algebras as extensions of two abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$.

Remark 4.1. Any two-step solvable Lie algebra $\mathfrak{g}$ is described by the following data: there is an exact sequence of Lie algebras:

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0$$

such that: $\mathfrak{a}$ and $\mathfrak{b}$ are abelian Lie algebras, $\varphi : \mathfrak{b} \to \mathrm{End}(\mathfrak{a})$ is a Lie algebra representation, $\Omega \in Z^2(\mathfrak{b}, \mathfrak{a})$ is a two-cocycle, and the Lie bracket of $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ is given by:

$$[(a, x), (b, y)] := (\varphi(x)b - \varphi(y)a + \Omega(x, y), 0).$$

Suppose we are given such an extension. Then $\mathfrak{a}$ and $\mathfrak{b}$ are vector spaces with trivial Lie brackets. In this case the Lie algebra $\mathfrak{g}$ is clearly two-step solvable. The conditions on φ and Ω reduce as follows: since $\mathfrak{a}$ and $\mathfrak{b}$ are abelian, φ is just a linear map satisfying:

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$$

for all $x, y \in \mathfrak{b}$. On the other hand, $\Omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}$ is a skew-symmetric bilinear map satisfying:

$$\varphi(x)\Omega(y, z) - \varphi(y)\Omega(x, z) + \varphi(z)\Omega(x, y) = 0.$$

Now we want to construct affine structures on Lie algebras $\mathfrak{g}$ which are an extension of a Lie algebra $\mathfrak{b}$ by an abelian Lie algebra $\mathfrak{a}$. Assume that $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ is an extension with the above data. Suppose that we have already an LSA-product $(a, b) \mapsto a \cdot b$ on $\mathfrak{a}$ and an LSA-product $(x, y) \mapsto x \cdot y$ on $\mathfrak{b}$. We want to lift these LSA-products to $\mathfrak{g}$. Consider:

$$\omega : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}, \qquad \varphi_1, \varphi_2 : \mathfrak{b} \to \mathrm{End}(\mathfrak{a})$$

where ω is a bilinear map and φ_1, φ_2 are Lie algebra representations. We will define a bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by:

$$(a, x) \circ (b, y) := (a \cdot b + \varphi_1(y)a + \varphi_2(x)b + \omega(x, y), x \cdot y)$$

(12)

Proposition 4.2. *The above product defines a left-symmetric structure on $\mathfrak{g}$ if and only if the following conditions hold*:

$$\omega(x, y) - \omega(y, x) = \Omega(x, y)$$

(13)

$$\varphi_2(x) - \varphi_1(x) = \varphi(x)$$

(14)

$$\varphi_2(x)\omega(y, z) - \varphi_2(y)\omega(x, z) - \varphi_1(z)\Omega(x, y) = \omega(y, x \cdot z) - \omega(x, y \cdot z) + \omega([x, y], z)$$

(15)

$$a \cdot \omega(y, z) + \varphi_1(y \cdot z)a = \varphi_2(y)\varphi_1(z)a - \varphi_1(z)\varphi(y)a$$

(16)

$$a \cdot (\varphi_1(z)b) = b \cdot (\varphi_1(z)a) \tag{17}$$

$$\varphi_2(y)(a \cdot c) - a \cdot (\varphi_2(y)c) = (\varphi(y)a) \cdot c \tag{18}$$

$$\Omega(x, y) \cdot c = 0 \tag{19}$$

for all $a, b, c \in \mathfrak{a}$ and $x, y, z \in \mathfrak{b}$.

Proof. Let $u = (a, x)$, $v = (b, y)$, $w = (c, z)$ denote three arbitrary elements of $\mathfrak{g}$. Let us first consider the Eq. (3) for the product, i.e. $[u, v] = u \circ v - v \circ u$. Using (11) and (12) and the commutativity of the LSA-product in $\mathfrak{a}$ we obtain:

$$[u, v] = (\varphi(x)b - \varphi(y)a + \Omega(x, y), [x, y]),$$

$$u \circ v - v \circ u = ((\varphi_2(x) - \varphi_1(x))b - (\varphi_2(y) - \varphi_1(y))a + \omega(x, y) - \omega(y, x), [x, y]).$$

Suppose that the two expressions are equal for all $a, b \in \mathfrak{a}$ and $x, y \in \mathfrak{b}$. For $a = b = 0$ we obtain $\omega(x, y) - \omega(y, x) = \Omega(x, y)$. Taking this into account, $a = 0$ implies $\varphi_2(x) - \varphi_1(x) = \varphi(x)$. Conversely, these two conditions imply (3).

The computation required for the left-symmetric condition (1) is a bit longer:

$$u \circ (v \circ w) - v \circ (u \circ w)$$

$$= (a \cdot (b \cdot c) - b \cdot (a \cdot c) + a \cdot (\varphi_1(z)b) - b \cdot (\varphi_1(z)a) + a \cdot (\varphi_2(y)c) - b \cdot (\varphi_2(x)c)$$

$$+ a \cdot \omega(y, z) - b \cdot \omega(x, z) + \varphi_1(y \cdot z)a - \varphi_1(x \cdot z)b + \varphi_2(x)(b \cdot c) - \varphi_2(y)(a \cdot c)$$

$$+ \varphi_2(x)\varphi_1(z)b - \varphi_2(y)\varphi_1(z)a + [\varphi_2(x), \varphi_2(y)]c + \varphi_2(x)\omega(y, z) - \varphi_2(y)\omega(x, z)$$

$$+ \omega(x, y \cdot z) - \omega(y, x \cdot z), x \cdot (y \cdot z) - y \cdot (x \cdot z)),$$

$$[u, v] \circ w = ((\varphi(x)b) \cdot c - (\varphi(y)a) \cdot c + \Omega(x, y) \cdot c + \varphi_1(z)\varphi(x)b - \varphi_1(z)\varphi(y)a$$

$$+ \varphi_1(z)\Omega(x, y) + \varphi_2([x, y])c + \omega([x, y], z), [x, y] \cdot z)$$

The product is left-symmetric, i.e. condition (1) is satisfied, if and only if these two expressions are always equal. For $a = b = c = 0$ this yields condition (15). Using this, and setting $a = b = 0$, one obtains (19). Again using this and letting $b = c = 0$, one obtains (16). Taking $z = x = b = 0$ yields (18). Finally the choice of $c = x = y = 0$ gives (17). Conversely, the above conditions ensure that condition (1) is satisfied. $\square$

Proposition 4.3. *The product (12) defines a Novikov structure on $\mathfrak{g}$ if and only if the conditions for a left-symmetric structure are satisfied, and in addition the following conditions hold*:

$$\varphi_1(z)\omega(x, y) - \varphi_1(y)\omega(x, z) = \omega(x \cdot z, y) - \omega(x \cdot y, z) \tag{20}$$

$$\omega(x, y) \cdot c + \varphi_2(x \cdot y)c = \varphi_1(y)\varphi_2(x)c \tag{21}$$

$$[\varphi_1(x), \varphi_1(y)] = 0 \tag{22}$$

$$(\varphi_2(x)b) \cdot c = (\varphi_2(x)c) \cdot b \tag{23}$$

$$\varphi_1(z)(a \cdot b) = (\varphi_1(z)a) \cdot b \tag{24}$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{25}$$

Proof. Since the product on $\mathfrak{a}$ is associative and commutative we have automatically:

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b \tag{26}$$

Let $u = (a, x)$, $v = (b, y)$, $w = (c, z)$ as before. We have:

$$(u \circ v) \circ w = ((a \cdot b) \cdot c + (\varphi_1(y)a) \cdot c + (\varphi_2(x)b) \cdot c + \omega(x, y) \cdot c$$
$$+ \varphi_1(z)(a \cdot b) + \varphi_1(z)\varphi_1(y)a + \varphi_1(z)\varphi_2(x)b + \varphi_1(z)\omega(x, y)$$
$$+ \varphi_2(x \cdot y)c + \omega(x \cdot y, z), (x \cdot y) \cdot z),$$

$$(u \circ w) \circ v = ((a \cdot c) \cdot b + (\varphi_1(z)a) \cdot b + (\varphi_2(x)c) \cdot b + \omega(x, z) \cdot b$$
$$+ \varphi_1(y)(a \cdot c) + \varphi_1(y)\varphi_1(z)a + \varphi_1(y)\varphi_2(x)c$$
$$+ \varphi_1(y)\omega(x, z) + \varphi_2(x \cdot z)b + \omega(x \cdot z, y), (x \cdot z) \cdot y)$$

Suppose that the product is Novikov, i.e. suppose that $(u \circ v) \circ w = (u \circ w) \circ v$. Then for $a = b = c = 0$ we obtain the conditions (20) and (25). For $x = y = z = 0$ we obtain (26), and for $a = b = 0$ it follows (21). For $b = c = 0$ it follows (22) and for $y = z = a = 0$ we obtain (23). Finally setting $x = y = c = 0$ yields (24). Conversely, the above conditions ensure that the product is Novikov. $\square$

If $\mathfrak{b}$ is also abelian, then the Lie algebra $\mathfrak{g}$ is two-step solvable. We say that the LSA-products on $\mathfrak{a}$ and $\mathfrak{b}$ are trivial if $a \cdot b = x \cdot y = 0$ for all $a, b \in \mathfrak{a}$ and $x, y \in \mathfrak{b}$.

Corollary 4.4. *Suppose that the LSA-products on $\mathfrak{a}$ and $\mathfrak{b}$ are trivial. Hence $\mathfrak{b}$ is also abelian. Then* (12) *defines a left-symmetric structure on $\mathfrak{g}$ if and only if the following conditions hold*:

$$\omega(x, y) - \omega(y, x) = \Omega(x, y), \qquad \varphi_2(x) - \varphi_1(x) = \varphi(x),$$

$$\varphi_2(x)\omega(y, z) - \varphi_2(y)\omega(x, z) = \varphi_1(z)\Omega(x, y), \qquad [\varphi_1(x), \varphi_2(y)] = \varphi_1(x)\varphi_1(y)$$

It defines a Novikov structure on $\mathfrak{g}$ if in addition:

$$\varphi_1(z)\omega(x, y) = \varphi_1(y)\omega(x, z), \qquad \varphi_1(x)\varphi_2(y) = 0, \qquad [\varphi_1(x), \varphi_1(y)] = 0$$

In particular, if $\varphi_1 = 0$, the product defines a Novikov structure on $\mathfrak{g}$ if and only if it defines a left-symmetric structure on $\mathfrak{g}$.

Corollary 4.5. *Assume that $\mathfrak{g} = \mathfrak{a} \rtimes_\varphi \mathfrak{b}$ is a semidirect product of an abelian Lie algebra $\mathfrak{a}$ and a Lie algebra $\mathfrak{b}$ by a representation $\varphi : \mathfrak{b} \to \mathrm{End}(\mathfrak{a}) = \mathrm{Der}(\mathfrak{a})$. This yields a split exact sequence*:

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{b} \to 0$$

If $\mathfrak{b}$ admits an LSA-product then also $\mathfrak{g}$ admits an LSA-product. If $\mathfrak{b}$ admits a Novikov product $(x, y) \mapsto x \cdot y$ such that:

$$\varphi(x \cdot y) = 0$$

for all $x, y \in \mathfrak{b}$ then also $\mathfrak{g}$ admits a Novikov product.

Proof. Because the short exact sequence is split, the two-cocycle Ω in the Lie bracket of $\mathfrak{g}$ is trivial, i.e. $\Omega(x, y) = 0$. Let $a \cdot b = 0$ be the trivial product on $\mathfrak{a}$ and take $\varphi_1 = 0$, $\omega(x, y) = 0$. Assume that $(x, y) \mapsto x \cdot y$ is an LSA-product. Then all conditions of Proposition 4.2 are satisfied. Hence (12) defines an LSA-product on $\mathfrak{g}$, given by:

$$(a, x) \circ (b, y) = (\varphi(x)b, x \cdot y).$$

Assume that the product on $\mathfrak{b}$ is Novikov. Then the product on $\mathfrak{g}$ will be Novikov if and only if the conditions of Proposition 4.3 are satisfied. In this case, only condition (21) remains, which says $\varphi(x \cdot y) = 0$. $\square$

Example 4.6. Let $\mathfrak{g}$ be the five-dimensional Lie algebra with basis (A, B, C, X, Y) and Lie brackets $[X, Y] = A, [X, A] = B, [Y, A] = C$. This is a free three-step nilpotent Lie algebra. Hence it is two-step solvable. The product (12) defines a Novikov structure on $\mathfrak{g}$ with:

$$\varphi_1(X) = \begin{pmatrix} 0 & 0 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \varphi_2(X) = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \varphi_2(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\varphi_1(Y) = 0$, $\omega(Y, X) = -A$, $\omega(X, X) = \omega(X, Y) = \omega(Y, Y) = 0$, and trivial products on $\mathfrak{a}$ and $\mathfrak{b}$.

The Lie algebra $\mathfrak{g}$ is given by $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ with $\mathfrak{a} = \mathrm{span}\{A, B, C\}$, $\mathfrak{b} = \mathrm{span}\{X, Y\}$ abelian, $\Omega(X, Y) = A$ and $\varphi(X)A = B$, $\varphi(Y)A = C$, i.e.:

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \varphi(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the conditions of Corollary 4.4 are satisfied. Note that any product $\varphi_i(x)\varphi_j(y) = 0$ for $1 \le i, j \le 2$ and $x, y \in \mathfrak{b}$. We can write down the resulting Novikov structure on $\mathfrak{g}$ explicitly. It is given by (12); we only write the non-zero products:

$$A \circ X = -B/2, \qquad X \circ A = B/2, \qquad Y \circ A = C, \qquad Y \circ X = -A.$$

Lemma 4.7. *Let $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ be a two-step solvable Lie algebra equipped with a left-symmetric structure comming from (12). Then $\mathfrak{a}$ is a two-sided ideal in the left-symmetric algebra $\mathfrak{g}$.*

Proof. $\mathfrak{a}$ is isomorphic to $\{(a, 0) \in \mathfrak{a} \times \mathfrak{b} = \mathfrak{g} | a \in \mathfrak{a}\}$. We have:

$$(a, 0) \circ (b, y) = (a \cdot b + \varphi_1(y)a, 0) \in \mathfrak{a} \quad \text{and} \quad (b, y) \circ (a, 0) = (b \cdot a + \varphi_2(y)a, 0) \in \mathfrak{a}.$$

$\square$

Remark 4.8. The lemma shows that there are two-step solvable, non-nilpotent Lie algebras admitting left-symmetric structures which do not arise by (12). To see this take, for example, any simple LSA with a two-step solvable Lie algebra. Note however, that the Lie algebra of a simple LSA cannot be nilpotent. There are many examples, see [6], i.e. consider for any $n \ge 2$ the algebra:

$$I_n = \langle e_1, \ldots, e_n | e_1 \circ e_1 = 2e_1, \quad e_1 \circ e_j = e_j, \quad e_j \circ e_j = e_1, \quad j \ge 2 \rangle.$$

This is a simple LSA. Hence it does not admit a non-trivial two-sided ideal $\mathfrak{a}$. Its Lie algebra $\mathfrak{g}$ is given by $[e_1, e_j] = e_j$ for $j = 2, 3, \ldots, n$. It is two-step solvable and the LSA-structure on it cannot arise by (12). Note that, of course, it does nevertheless admit some Novikov structure. Just consider the structure given by $e_1 \circ e_j = e_j$ for all $j \ge 2$, and the other products equal to zero.

Proposition 4.9. *Let $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ be a two-step solvable Lie algebra. If there exists an $e \in \mathfrak{b}$ such that $\varphi(e) \in \mathrm{End}(\mathfrak{a})$ is an isomorphism, then $\mathfrak{g}$ admits a Novikov structure. In fact, in that case (12) defines a Novikov product, where $\varphi_1 = 0$, $\varphi_2 = \varphi$, the product on $\mathfrak{a}$ and $\mathfrak{b}$ is trivial, and:*

$$\omega(x, y) = \varphi(e)^{-1}\varphi(x)\Omega(e, y).$$

Proof. We have to show that the conditions of Corollary 4.4 are satisfied. Applying $\varphi(e)^{-1}$ to (10) with $z = e$ it follows $\Omega(x, y) - \varphi(e)^{-1}\varphi(x)\Omega(e, y) + \varphi(e)^{-1}\varphi(y)\Omega(e, x) = 0$. This just means that $\Omega(x, y) = \omega(x, y) - \omega(y, x)$. Furthermore we have:

$$\varphi(x)\omega(y, z) - \varphi(y)\omega(x, z) = \varphi(x)\varphi(e)^{-1}\varphi(y)\Omega(e, z) - \varphi(y)\varphi(e)^{-1}\varphi(x)\Omega(e, z)$$

$$= \varphi(e)^{-1}(\varphi(x)\varphi(y) - \varphi(y)\varphi(x))\Omega(e, z) = 0.$$

Hence the product defines a left-symmetric structure. Since $\varphi_1 = 0$ the structure is also Novikov.

We can apply our results to give a new proof of Scheuneman's result [10]:

Proposition 4.10. *Any three-step nilpotent Lie algebra admits a LSA structure.*

Proof. Let $\mathfrak{g}$ be a three-step nilpotent Lie algebra. Since $\mathfrak{g}$ is two-step solvable, we can use Proposition 4.2. We will show the existence of a left-symmetric structure on $\mathfrak{g}$ as follows. Let $(e_1, \ldots, e_m)$ be a basis of $\mathfrak{b}$, $(f_1, \ldots, f_n)$ be a basis of $\mathfrak{a}$ and introduce the following notation:

$$X_i = \varphi_1(e_i), \qquad Y_i = \varphi_2(e_i), \qquad A_i = \varphi(e_i), \qquad x_{ij} = \omega(e_i, e_j), \qquad v_{ij} = \Omega(e_i, e_j)$$

for $1 \leq i, j \leq m$. The Lie algebra $\mathfrak{g}$ is determined by the data $\mathfrak{a}, \mathfrak{b}, A_i, v_{ij}$ satisfying

$$[A_i, A_j] = 0, \qquad v_{ji} + v_{ij} = 0, \qquad A_i v_{jk} - A_j v_{ik} + A_k v_{ij} = 0.$$

We have $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{b} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. The operators $A_i, A_j \in \text{End}(\mathfrak{a})$ satisfy $A_i A_j([e_k, e_l]) = [e_i, [e_j, [e_k, e_l]]] = 0$ since $\mathfrak{g}$ is three-step nilpotent. Thus we have:

$$A_i A_j = 0.$$

Let $Y_i := X_i + A_i$. We have to find $x_{ij} \in \mathfrak{a}$ and X_i such that:

$$Y_i x_{jk} - Y_j x_{ik} - X_k v_{ij} = 0, \qquad x_{ij} - x_{ji} = v_{ij}, \qquad [X_i, A_j] = X_j X_i$$

for all i, j. A possible solution is given by:

$$x_{ij} = \tfrac{1}{2} v_{ij}, \qquad X_i = -\tfrac{1}{3} A_i, \qquad Y_i = \tfrac{2}{3} A_i.$$

Then the first equation is just $\frac{1}{3}(A_i v_{jk} - A_j v_{ik} + A_k v_{ij}) = 0$, which is true by assumption. The third equation is trivially true since $A_j A_i = 0$ for all i, j. $\quad\square$

Remark 4.11. It is natural to ask whether the above result holds also for Novikov structures. We will show that all three-step nilpotent Lie algebras with two or three generators admit a Novikov structure. However, a Novikov structure may not exist in general on a three-step nilpotent Lie algebra. Unfortunately it seems to be hard to find a counter-example because of dimension reasons. We will later generalize Scheuneman's result, see Proposition 5.7.

Suppose that $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ is a two-step solvable Lie algebra with the notations as above. The assumptions on A_i and v_{ij} for $1 \leq i, j \leq m$ are as follows, see (9) and (10):

$$v_{ij} + v_{ji} = 0 \tag{27}$$

$$A_i A_j - A_j A_i = 0 \tag{28}$$

$$A_i v_{jk} - A_j v_{ik} + A_k v_{ij} = 0 \tag{29}$$

If we assume that the products on $\mathfrak{a}$ and $\mathfrak{b}$ are trivial, then the conditions of Corollary 4.4 mean that we have to find, for a Novikov structure, X_i and x_{ij} such that, for all i, j, k:

$$x_{ij} - x_{ji} = v_{ij} \tag{30}$$

$$X_i + A_i = Y_i \tag{31}$$

$$Y_i x_{jk} - Y_j x_{ik} = X_k v_{ij} \tag{32}$$

$$X_i A_j - A_j X_i = X_j X_i \tag{33}$$

$$X_k x_{ij} = X_j x_{ik} \tag{34}$$

$$X_i Y_j = 0 \tag{35}$$

$$X_i X_j = X_j X_i \tag{36}$$

Note that (28), (31) and (33) imply:

$$Y_i Y_j = Y_j Y_i.$$

Indeed:

$$[Y_i, Y_j] = [X_i + A_i, X_j + A_j] = [X_i, X_j] + [X_i, A_j] + [A_i, X_j] + [A_i, A_j]$$
$$= [X_i, X_j] + X_j X_i - X_i X_j = 0.$$

Proposition 4.12. *Let $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ be a two-generated three-step nilpotent Lie algebra, i.e., dim $\mathfrak{b} = 2$. Then $\mathfrak{g}$ admits a Novikov structure.*

Proof. A two-generated three-step nilpotent Lie algebra is a quotient of the five-dimensional free nilpotent Lie algebra of Example 4.6. It would not be difficult to verify directly that such an algebra admits a Novikov structure. However in this case we can simply use Corollary 4.4 to construct a Novikov structure. Let $\mathfrak{b} = \langle e_1, e_2 \rangle$. Note that the assumption (29) is automatically satisfied since dim $\mathfrak{b} = 2$: there are always two indices equal, so that any skew-symmetric map Ω is a two-cocycle. We have to find $X_1, X_2 \in \mathrm{End}(\mathfrak{a})$ and $x_{ij} \in \mathfrak{a}$ such that the conditions (30), ..., (36) hold. It is easy to see that a solution is given by:

$$X_1 = -A_1/2, \qquad X_2 = 0, \qquad x_{11} = 0, \qquad x_{12} = 0, \qquad x_{21} = -v_{12}, \qquad x_{22} = 0.$$

$\square$

Example 4.13. Let $\mathfrak{f}$ be the free three-step nilpotent Lie algebra with three generators. Then $\mathfrak{f}$ admits a Novikov structure.

Let $(x_1, \ldots, x_{14})$ be a basis of $\mathfrak{f}$ with generators (x_1, x_2, x_3) and Lie brackets

$$x_4 = [x_1, x_2], \qquad x_5 = [x_1, x_3], \qquad x_6 = [x_2, x_3], \qquad x_7 = [x_1, [x_1, x_2]] = [x_1, x_4],$$

$$x_8 = [x_2, [x_1, x_2]] = [x_2, x_4], \qquad x_9 = [x_3, [x_1, x_2]] = [x_3, x_4],$$

$$x_{10} = [x_1, [x_1, x_3]] = [x_1, x_5], \qquad x_{11} = [x_2, [x_1, x_3]] = [x_2, x_5],$$

$$x_{12} = [x_3, [x_1, x_3]] = [x_3, x_5], \qquad x_{11} - x_9 = [x_1, [x_2, x_3]] = [x_1, x_6],$$

$$x_{13} = [x_2, [x_2, x_3]] = [x_2, x_6], \qquad x_{14} = [x_3, [x_2, x_3]] = [x_3, x_6]$$

The Jacobi identity is satisfied. A Novikov structure is given by:

$$x_1 \circ x_3 = x_5, \qquad x_1 \circ x_4 = \tfrac{1}{2}x_7, \qquad x_1 \circ x_5 = x_{10}, \qquad x_1 \circ x_6 = x_{11} - \tfrac{1}{2}x_9,$$

$$x_2 \circ x_1 = -x_4, \qquad x_2 \circ x_3 = x_6, \qquad x_2 \circ x_4 = x_8, \qquad x_2 \circ x_5 = x_{11},$$

$$x_2 \circ x_6 = x_{13}, \qquad x_3 \circ x_4 = \tfrac{1}{2}x_9, \qquad x_3 \circ x_5 = \tfrac{1}{2}x_{12}, \qquad x_3 \circ x_6 = \tfrac{1}{2}x_{14},$$

$$x_4 \circ x_1 = -\tfrac{1}{2}x_7, \qquad x_4 \circ x_3 = -\tfrac{1}{2}x_9, \qquad x_5 \circ x_3 = -\tfrac{1}{2}x_{12},$$

$$x_6 \circ x_1 = \tfrac{1}{2}x_9, \qquad x_6 \circ x_3 = -\tfrac{1}{2}x_{14}.$$

Proposition 4.14. *Any three-generated three-step nilpotent Lie algebra admits a Novikov structure.*

Proof. Let $\mathfrak{g}$ be a three-generated three-step nilpotent Lie algebra. Then $\mathfrak{g}$ is isomorphic to a quotient $\mathfrak{f}/I$, where $\mathfrak{f}$ denotes the free three-step nilpotent Lie algebra on three generators and I is a Lie algebra ideal in $\mathfrak{f}$. In the example above we showed that $\mathfrak{f}$ admits a Novikov structure. To prove this proposition it is enough to show that we can choose I in such a way that it is also a two-sided ideal for the Novikov-product. For in this situation, the product on $\mathfrak{f}$ is inherited by $\mathfrak{g}$. We can distinguish three cases:

(1) dim $\mathfrak{g}^2/\mathfrak{g}^3 = 3$: In this case $I \subseteq Z(\mathfrak{f})$ and it is obvious that I is an ideal for the Novikov product.
(2) dim $\mathfrak{g}^2/\mathfrak{g}^3 = 2$: This is only possible if $I = \langle c + z \rangle \oplus Z$, where $c \in \mathfrak{f}^2$, $z \in \mathfrak{f}^3 = Z(\mathfrak{f})$ and Z is a subspace of $Z(\mathfrak{f})$. As $\mathfrak{f}$ is generated by 3 elements, any element of $\mathfrak{f}^2$ can be written as a genuine Lie bracket $c = [u, v]$ for some $u, v \in \mathfrak{f}$ (and not only as a linear combination of Lie brackets). Therefore, we can without loss of generality assume that, with the notations from the example above, $c = [x_1, x_2] = x_4$. Since I is a Lie algebra ideal, we obtain that $[x_1, c + z] = [x_1, x_4] = x_7 \in I$, $[x_2, c + z] = x_8 \in I$ and $[x_3, c + z] = x_9 \in I$. It follows that $I = \langle x_4 + z, x_7, x_8, x_9 \rangle \oplus Z'$, with $Z' \subseteq \mathfrak{f}^3$. It is now obvious again that I is actually a Novikov ideal.
(3) dim $\mathfrak{g}^2/\mathfrak{g}^3 = 1$: Analogously as in the previous case, we can assume that $I = \langle x_4 + z_1, x_5 + z_2 \rangle \oplus Z$, where $z_1, z_2 \in \mathfrak{f}^3$ and $Z \subseteq \mathfrak{f}^3$ and continue exactly as before. $\square$

Let $J(n)$ denote the full Jordan block of size n to the eigenvalue zero, i.e.:

$$J(n) = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Proposition 4.15. *Let $\mathfrak{g} = (\mathfrak{a}, \mathfrak{b}, \varphi, \Omega)$ be a two-step solvable Lie algebra. If there exists an $x \in \mathfrak{b}$ such that the Jordan form of $\varphi(x) \in \mathrm{End}(\mathfrak{a})$ is $J(n)$, where $\dim \mathfrak{a} = n$, then $\mathfrak{g}$ admits a Novikov structure.*

Proof. We may assume that $(e_1, \dots, e_m)$ is a basis for $\mathfrak{b}$ such that $A_1 = \varphi(e_1) = J(n)$. Since all A_i commute with A_1 by (28) we have:

$$A_i = \gamma_{i,0}E + \gamma_{i,1}A_1 + \gamma_{i,2}A_1^2 + \dots + \gamma_{i,n-1}A_1^{n-1} \tag{37}$$

for $\gamma_{i,j} \in k$. If $\gamma_{i,0} \neq 0$ for some i then it follows $\det A_i \neq 0$. Then $\mathfrak{g}$ admits a Novikov structure by Proposition 4.9. Hence we may assume $\gamma_{i,0} = 0$ for all i. But then we may also assume that $\gamma_{i,1} = 0$ for $i = 2, \dots, m$. Just consider the new basis $(e_1, e_2 - \gamma_{2,1}e_1, \dots, e_m - \gamma_{m,1}e_1)$ for $\mathfrak{b}$.

Now we claim that (12) defines a Novikov product, where $\varphi_1 = 0$, $\varphi_2 = \varphi$, the product on $\mathfrak{a}$ and $\mathfrak{b}$ is trivial, $A_1 = \varphi(e_1) = J(n)$ and

$$X_i = 0, \ i = 1, \ldots, m, \qquad x_{11} = 0, \qquad x_{1j} = v_{1j}, \ j = 2, \ldots, m,$$

$$x_{ij} = A_1^t A_i v_{1j}, \ i = 2, \ldots, m, \ i \leq j, \qquad x_{ji} = x_{ij} - v_{ij}, \ j > I$$

We first show that

$$A_1 A_1^t A_j = A_j \tag{38}$$

$$A_i A_1^t A_j = A_j A_1^t A_i \tag{39}$$

for all i, j: we have $A_1 A_1^t A_1 = A_1$ since $A_1 = J(n)$. It follows $A_1 A_1^t A_1^k = A_1^k$ for all $k \geq 1$. Since each A_j is a polynomial in A_1 the first claim follows. Similarly $A_i A_1^t A_1 = A_i$ so that:

$$A_i A_1^t A_1^k = A_i A_1^{k-1} = A_1^{k-1} A_i = A_1^k A_1^t A_i.$$

Since each A_j is a polynomial in A_1 the second claim follows. Recall that we assume (29). Since $X_i = 0$ for all i we only have to show that (32) is satisfied:

$$A_i x_{jk} = A_j x_{ik}$$

for all i, j, k in $[1, m]$. According to the definition of x_{ij} we have to consider different cases. Let us first assume that i, j, $k \geq 2$. The first case is $i \leq k$, $j \leq k$. We have to show:

$$A_i A_1^t A_j v_{1k} = A_j A_1^t A_i v_{1k},$$

which follows from (39). The second case is $i \leq k$, $j > k$. We have to show:

$$A_i (A_1^t A_k v_{1j} - v_{kj}) = A_j A_1^t A_i v_{1k}.$$

Using (39) and (29) we obtain:

$$0 = (A_i A_1^t)(A_k v_{1j} + A_1 v_{jk} - A_j v_{1k}) = A_i A_1^t A_k v_{1j} - A_i A_1^t A_1 v_{kj} - A_i A_1^t A_j v_{1k}$$

$$= A_i A_1^t A_k v_{1j} - A_1 A_1^t A_i v_{kj} - A_j A_1^t A_i v_{1k} = A_i (A_1^t A_k v_{1j} - v_{kj}) - A_j A_1^t A_i v_{1k}.$$

The third case is $k < i$, $j > k$. Then we have to show:

$$A_i (A_1^t A_k v_{1j} - v_{kj}) = A_j (A_1^t A_k v_{1i} - v_{ki}).$$

We have:

$$0 = (A_j A_1^t)(A_i v_{1k} - A_k v_{1i} + A_1 v_{ki}) = A_j A_1^t A_i v_{1k} - A_j A_1^t A_k v_{1i} + A_j A_1^t A_1 v_{ki}$$

$$= A_i A_1^t (A_j v_{1k}) - A_j A_1^t A_k v_{1i} + A_j v_{ki} = A_i A_1^t (A_k v_{1j} - A_1 v_{kj}) - A_j A_1^t A_k v_{1i} + A_j v_{ki}$$

$$= A_i (A_1^t A_k v_{1j} - v_{kj}) - A_j (A_1^t A_k v_{1i} - v_{ki}).$$

Finally suppose that one of the indices is equal to one. Assume that $i = 1$. If $j \leq k$ we have to show:

$$A_1 (A_1^t A_j v_{1k}) = A_j v_{1k},$$

which follows from (38). If $j > k \geq 2$ we have to show

$$A_1 (A_1^t A_k v_{1j} - v_{kj}) = A_j v_{1k},$$

which follows from (29). For $k = i = 1$ there is nothing to show. Similarly the other cases are shown. $\square$

Corollary 4.16. *Any filiform Lie algebra with abelian commutator algebra admits a Novikov structure.*

5. A reduction to the case of nilpotent Lie algebras

In the previous section, we established some examples of LSA and Novikov structures on several classes of Lie algebras. All of these examples were nilpotent. One might think that these cases are too special and that we should look for solvable, but non-nilpotent Lie algebras. However, in this section we will show that the lifting procedure can in many cases be reduced to the nilpotent case.

To obtain this result, we need some preliminary work on the structure of modules over a nilpotent Lie algebra.

Lemma 5.1. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over a field k of characteristic 0, equipped with a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(n, k)$. Then $k^{n \times 1}$ (resp. $k^{1 \times n}$) becomes a $\mathfrak{g}$-module, via $X \cdot v = \varphi(X)v$, $\forall X \in \mathfrak{g}, \forall v \in k^{n \times 1}$ (resp. $X \cdot v = -v\varphi(X)$, $\forall X \in \mathfrak{g}, \forall v \in k^{1 \times n}$). We then have that, for these module structures:*

$$H^0(\mathfrak{g}, k^{n \times 1}) = 0 \iff H^0(\mathfrak{g}, k^{1 \times n}) = 0$$

Proof. Recall that $H^0(\mathfrak{g}, M)$ is the space $M^{\mathfrak{g}} = \{m \in M | X \cdot m = 0, \ \forall X \in \mathfrak{g}\}$. Suppose first that $H^0(\mathfrak{g}, k^{n \times 1}) = 0$ and assume that there exists a non-zero $w \in k^{1 \times n}$ for which $-w\varphi(X) = 0$. Then, after conjugating with an invertible $n \times n$-matrix, we may suppose that:

$$\varphi(X) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ * & & & & \\ * & & & & \\ \vdots & & & * & \\ * & & & & \end{pmatrix}, \qquad \forall X \in \mathfrak{g}.$$

Now, let $V \subseteq k^{n \times 1}$ be the subspace consisting of all column-vectors $v = (0, a_2, a_3, \ldots, a_n)^t$ having 0 as their first coordinate. It is obvious that V is a $\mathfrak{g}$-submodule of $k^{n \times 1}$ and there is a short exact sequence of $\mathfrak{g}$-modules:

$$0 \to V \to k^{n \times 1} \to k \to 0$$

where k stands for the trivial one-dimensional $\mathfrak{g}$-module. This short exact sequence gives rise to a long exact sequence in cohomology:

$$0 \to H^0(\mathfrak{g}, V) \to H^0(\mathfrak{g}, k^{n \times 1}) \to H^0(\mathfrak{g}, k) \to H^1(\mathfrak{g}, V) \to \cdots$$

As $H^0(\mathfrak{g}, k^{n \times 1}) = 0$, we also have that $H^0(\mathfrak{g}, V) = 0$. Because $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra, this implies that $H^i(\mathfrak{g}, V) = 0$, for all $i \geq 0$ (see [7]). It follows that in the exact sequence above, $H^0(\mathfrak{g}, k)$ is standing between two 0-terms and hence must itself be 0. But this contradicts the fact that for the trivial module, we have that $H^0(\mathfrak{g}, k) = k$.

The other direction can be proved in a similar way. $\square$

Let $\mathfrak{g}$ be a Lie algebra over a field k and assume that $\varphi_1 : \mathfrak{g} \to \mathfrak{gl}(n_1, k)$ and $\varphi_2 : \mathfrak{g} \to \mathfrak{gl}(n_2, k)$ are two Lie algebra representations. Then $k^{n_1 \times n_2}$ becomes a $\mathfrak{g}$–module via the map:

$$\varphi : \mathfrak{g} \to \mathrm{End}(k^{n_1 \times n_2}) : X \mapsto \varphi(X)$$

where

$$\forall B \in k^{n_1 \times n_2} : \quad \varphi(X)(B) = \varphi_1(X)B - B\varphi_2(X).$$

We refer to φ as being the *combination* of φ_1 and φ_2.

Lemma 5.2. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over a field k of characteristic 0. Assume that $\varphi_1 : \mathfrak{g} \to \mathfrak{gl}(n_1, k)$ and $\varphi_2 : \mathfrak{g} \to \mathfrak{gl}(n_2, k)$ are two Lie algebra representations, such that*:

(1) $\varphi_1(\mathfrak{g})$ *consists of nilpotent matrices, and*
(2) $H^0_{\varphi_2}(\mathfrak{g}, k^{n_2 \times 1}) = 0$.

If φ denotes the combination of φ_1 and φ_2, then:

$$H^i_\varphi(\mathfrak{g}, k^{n_1 \times n_2}) = 0, \quad \forall i \geq 0.$$

Proof. Note that by [7] again, we only need to prove the lemma for $i = 0$. We proceed by induction on n_1. If $n_1 = 1$, then $\varphi_1 \equiv 0$ and the claim follows from Lemma 5.1. Now, let $n_1 > 1$. Without loss of generality, we may assume that $\forall X \in \mathfrak{g} : \varphi_1(X)$ is an upper-triangular matrix with 0's on the diagonal. Let $B = (b_{i,j}) \in k^{n_1 \times n_2}$ be an element with:

$$0 = \varphi(X)B = \varphi_1(X)B - B\varphi_2(X), \quad \forall X \in \mathfrak{g}.$$

Looking at the last row of the above equality, we find:

$$(0, 0, \dots, 0) = (0, 0, \dots, 0) - (b_{n_1,1}, b_{n_1,2}, \dots, b_{n_1,n_2})\varphi_2(X), \quad \forall X \in \mathfrak{g}$$

From the case $n_1 = 1$ (or the previous lemma), we obtain that the last row of B has to be the zero row. The rest now follows by induction. $\quad \square$

Proposition 5.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field of characteristic 0 and let V be a finite-dimensional $\mathfrak{g}$-module. Then V can be written as a direct sum*:

$$V_n \oplus V_0,$$

where V_n is a nilpotent $\mathfrak{g}$-module (in fact the unique maximal nilpotent submodule of V) and V_0 is also a $\mathfrak{g}$-module with $H^i(\mathfrak{g}, V_0) = 0, \forall i \geq 0$.

Proof. It is easy to see that V contains a unique maximal nilpotent submodule V_n. Therefore, after choosing a suitable basis for V, we can assume that the module structure is given in matrix form $\psi : \mathfrak{g} \to \mathfrak{gl}(n_1 + n_2, k)$ ($n_1 + n_2$ is the dimension of V), with

$$\forall X \in \mathfrak{g} : \quad \psi(X) = \begin{pmatrix} \varphi_1(X) & B_X \\ 0 & \varphi_2(X) \end{pmatrix}$$

where $\varphi_1 : \mathfrak{g} \to \mathfrak{gl}(n_1, k)$ has images inside the set of upper triangular matrices with 0's on the diagonal and $\varphi_2 : \mathfrak{g} \to \mathfrak{gl}(n_2, k)$ is such that $H^0_{\varphi_2}(\mathfrak{g}, k^{n_2 \times 1}) = 0$ (and hence $H^i_{\varphi_2}(\mathfrak{g}, k^{n_2 \times 1}) = 0$ as before).

Let φ denote the combination of φ_1 and φ_2 as above. We claim that the map $B : \mathfrak{g} \to k^{n_1 \times n_2} : X \mapsto B_X$ is a one-cocycle (with respect to the module structure determined by φ). To see this, we must check that:

$$B_{[X,Y]} = \varphi(X)B_Y - \varphi(Y)B_X = \varphi_1(X)B_Y - B_Y\varphi_2(X) - \varphi_1(Y)B_X + B_X\varphi_2(Y).$$

This is easily seen to hold, by writing out the identity $\psi([X, Y]) = \psi(X)\psi(Y) - \psi(Y)\psi(X)$. By Lemma 5.2, we know that $H^1_\varphi(\mathfrak{g}, k^{n_1 \times n_2}) = 0$ and hence B has to be a coboundary. Therefore, there exists a $\alpha \in k^{n_1 \times n_2}$ such that:

$$B_X = \varphi(X)\alpha = \varphi_1(X)\alpha - \alpha\varphi_2(X).$$

If we now conjugate ψ with the matrix $\begin{pmatrix} I_{n_1} & \alpha \\ 0 & I_{n_2} \end{pmatrix}$, we get:

$$\begin{pmatrix} I_{n_1} & \alpha \\ 0 & I_{n_2} \end{pmatrix} \psi(X) \begin{pmatrix} I_{n_1} & -\alpha \\ 0 & I_{n_2} \end{pmatrix}$$

$$= \begin{pmatrix} I_{n_1} & \alpha \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} \varphi_1(X) & B_X \\ 0 & \varphi_2(X) \end{pmatrix} \begin{pmatrix} I_{n_1} & -\alpha \\ 0 & I_{n_2} \end{pmatrix} = \begin{pmatrix} \varphi_1(X) & 0 \\ 0 & \varphi_2(X) \end{pmatrix}.$$

This shows that we can decompose V into two submodules as claimed. $\quad\square$

We are now ready to apply our results to the lifting problem of LSA or Novikov structures. Let $\mathfrak{a}$ be an abelian Lie algebra as before and assume that $\mathfrak{b}$ is a nilpotent Lie algebra. Let $\varphi : \mathfrak{b} \to \operatorname{End}(\mathfrak{a})$ denote a $\mathfrak{b}$-module structure of $\mathfrak{a}$. By Proposition 5.3, we can write $\mathfrak{a}$ as a direct sum of $\mathfrak{b}$-modules $\mathfrak{a} = \mathfrak{a}_n \oplus \mathfrak{a}_0$, where $\mathfrak{a}_n$ is a nilpotent $\mathfrak{b}$-module and $H^0(\mathfrak{b}, \mathfrak{a}_0) = 0$. We can write $\varphi = \varphi' \oplus \varphi''$, where φ' (resp. φ'') is the restriction of φ to $\mathfrak{a}_n$ (resp. $\mathfrak{a}_0$). It follows that for any i, we have that:

$$H^i(\mathfrak{b}, \mathfrak{a}) = H^i(\mathfrak{b}, \mathfrak{a}_n) \oplus H^i(\mathfrak{b}, \mathfrak{a}_0) = H^i(\mathfrak{b}, \mathfrak{a}_n) \oplus 0 = H^i(\mathfrak{b}, \mathfrak{a}_n).$$

Any two-cocycle $\Omega \in Z^2(\mathfrak{b}, \mathfrak{a})$ is a sum of two two-cocycles:

$$\Omega = \Omega' + \Omega'', \quad \text{with} \quad \Omega' \in Z^2(\mathfrak{b}, \mathfrak{a}_n) \quad \text{and} \quad \Omega'' \in Z^2(\mathfrak{b}, \mathfrak{a}_0).$$

As $H^2(\mathfrak{b}, \mathfrak{a}_0) = 0$, we have that Ω is in fact cohomologuous to Ω'. Now, let $\mathfrak{g}$ denote the extension of $\mathfrak{b}$ by $\mathfrak{a}$ determined by the two-cocycle Ω. Without loss of generality we may assume that $\Omega'' = 0$. Note that Ω' determines an extension $\mathfrak{g}_n$ of $\mathfrak{b}$ by $\mathfrak{a}_n$ which is in fact given by:

$$0 \to \mathfrak{a}_n = \mathfrak{a}/\mathfrak{a}_0 \to \mathfrak{g}_n = \mathfrak{g}/\mathfrak{a}_0 \to \mathfrak{b} \to 0 \tag{40}$$

As $\mathfrak{b}$ is nilpotent and $\mathfrak{a}_n$ is a nilpotent $\mathfrak{b}$-module, we have that $\mathfrak{g}_n$ is a nilpotent Lie algebra. We will refer to the extension (40) as the nilpotent extension induced by the extension $\mathfrak{g}$. We can now formulate our reduction statement.

Proposition 5.4. *Let $\mathfrak{a}$ be an abelian Lie algebra equipped with a trivial LSA-product and let $\mathfrak{b}$ be a nilpotent Lie algebra equipped with a left-symmetric structure. Let $\mathfrak{g}$ denote an extension of $\mathfrak{b}$ by $\mathfrak{a}$ and let $\mathfrak{g}_n$ be the induced nilpotent extension. If the left symmetric-structure of $\mathfrak{b}$ lifts to the extension $\mathfrak{g}_n$, then it also lifts to $\mathfrak{g}$.*

Proof. As above, we can assume that $\mathfrak{g}$ is determined by a cocycle $\Omega = \Omega'$, with $\Omega(x, y) \in \mathfrak{a}_n, \forall x, y \in \mathfrak{b}$. The extension $\mathfrak{g}_n$ is represented by the same 2-cocycle, but then seen as an element of $Z^2(\mathfrak{b}, \mathfrak{a}_n)$.

We have to show that the conditions of Proposition 4.2 are satisfied for $\mathfrak{g}$, given they are satisfied for $\mathfrak{g}_n$. Let:

$$\omega' : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}_n \quad \text{and} \quad \varphi'_1, \varphi'_2 : \mathfrak{b} \to \operatorname{End}(\mathfrak{a}_n)$$

denote the maps needed to fulfil all the condition of Proposition 4.2 for the extension $\mathfrak{g}_n$. Now, let:

$$\omega'' : \mathfrak{b} \times \mathfrak{b} \to \mathfrak{a}_0 : (x, y) \mapsto 0 \quad \text{and} \quad \varphi_1'', \varphi_2'' : \mathfrak{b} \to \text{End}(\mathfrak{a}), \quad \text{with } \varphi_2'' = \varphi'' \text{ and } \varphi_1 \equiv 0.$$

It is now obvious that:

$$\omega = \omega' + \omega'', \qquad \varphi_1 = \varphi_1' + \varphi_1'' \quad \text{and} \quad \varphi_2 = \varphi_2' + \varphi_2''$$

satisfy all of the conditions of Proposition 4.2 for the extension $\mathfrak{g}$. $\quad\square$

Remark 5.5. This proposition provides an alternative proof for Proposition 4.9, because the conditions of this proposition immediately imply that $\mathfrak{a}_n = 0$, hence $\mathfrak{g}_n = \mathfrak{b}$.

Analoguously, we find.

Proposition 5.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be abelian Lie algebras equipped with a trivial LSA-product and let $\mathfrak{g}$ denote an extension of $\mathfrak{b}$ by $\mathfrak{a}$ with induced nilpotent extension $\mathfrak{g}_n$. If we can lift the trivial LSA-structures to a Novikov structure on $\mathfrak{g}_n$, then we can also lift it to a Novikov structure on $\mathfrak{g}$.*

Proof. The proof is analoguous to the previous proposition, but now we have to satisfy the conditions of Corollary 4.4. $\quad\square$

As an application of the reduction we obtained in this section, we prove the following generalization of Scheuneman's result.

Proposition 5.7. *Let $\mathfrak{g}$ be a two-step solvable Lie algebra with $\mathfrak{g}^r = \mathfrak{g}^4$ for all $r \geq 5$. Then $\mathfrak{g}$ admits a complete left symmetric structure.*

Proof. $\mathfrak{g}$ can be seen as an extension of abelian Lie algebras:

$$0 \to [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to 0.$$

As a $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$-module, we have a decomposition $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]_n \oplus [\mathfrak{g}, \mathfrak{g}]_0$ and the resulting quotient $\mathfrak{g}_n = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]_0$ is a nilpotent Lie algebra. Hence there exists a c such that $\mathfrak{g}^c \subseteq [\mathfrak{g}, \mathfrak{g}]_0$ and thus certainly $\mathfrak{g}^4 \subseteq [\mathfrak{g}, \mathfrak{g}]_0$. It follows that $\mathfrak{g}_n$ is nilpotent of class ≤ 3. By Proposition 4.10 and its proof, we know that the trivial LSA-structures on $\mathfrak{a}_n = [\mathfrak{g}, \mathfrak{g}]_n$ and $\mathfrak{b} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ lift to an LSA structure on the extension $0 \to \mathfrak{a}_n \to \mathfrak{g}_n \to \mathfrak{b} \to 0$. By Proposition 5.4, we can conclude that we also obtain an LSA-structure on $\mathfrak{g}$. $\quad\square$

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